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**Søgaard-Andersen, Per**

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## EIGENSTRUCTURE CONDITIONS FOR LOOP TRANSFER RECOVERY

Per Sogaard-Andersen

Control Engineering Institute, Technical University of Denmark

Bldg. 424 - DK-2800 Lyngby, Denmark

**Abstract**

Conditions for loop transfer recovery based on eigenstructure assignment are derived. These conditions imply constraints on the eigenvalues and certain parameter vectors of the controller. Three cases emerge - depending on the geometric structure of the plant. For all cases explicit design rules are derived. The eigenstructure-LTR approach provides - beyond insight into mechanisms of LTR - improved flexibility in selecting the controller gains and faster recovery convergence as compared with the LQG-based LTR method. These issues are illustrated in an example.

**1. Introduction.**

In the last decade a number of new tools for control system design with robustness specifications have emerged. One of the most useful has been the Linear-Quadratic-Gaussian procedure with Loop Transfer Recovery (LQG/LTR). In this procedure a target feedback loop (fullstate or observer) which reflects the robustness specifications is recovered with some suitable asymptotic design [S2].

In this paper the focus is on the second step of this procedure - the recovery step. Based on an eigenstructure interpretation of the recovery principle an alternative to the LQG-based LTR method is presented.

The motivation for working with this alternative method is that the eigenstructure approach provides the designer with more flexibility in selecting gains of minimal amplitude - while still satisfying the specifications - and faster convergence of the recovery process (which also produces smaller gains), beyond the new insight into the mechanisms of recovery that it provides.

Recently Kazerooni and Houpt [K1] has derived some results concerning eigenstructure-based LTR. However, as it is pointed out in [S5], these results only guarantee LTR in very special cases. In [S5] some improved results were outlined, but here certain assumptions on the geometric structure on the plant were imposed.

In this paper no such assumptions are imposed, and a more general approach to LTR based on eigenstructure assignment is presented. The new results are based on the analysis of the eigenstructure of highgain feedback systems presented in [S4]. The present paper does not discuss the theoretical background. Such a perspective can be found in [S4, S6].

The paper is organized as follows. In § 2 the robustness concepts underlying the LTR-approach are briefly discussed. In § 3 the eigenstructure equations are introduced, followed in § 4 by the outline of the eigenstructure-LTR results. In § 5 some remarks on the usefulness of this design-concept are given, and in § 6,7 some examples and concluding remarks are provided.

**2. The significance of loop transfer recovery.**

The loop transfer recovery concept is related to control system robustness via the recently developed singular value-based loopshaping paradigm. In this setting the robustness constraints are formulated as frequency-dependent bounds that the maximum singular value of the sensitivity and complementary sensitivity functions must satisfy:

$$\sigma_1[S(j\omega)] < p(\omega) \quad (2-1)$$

$$\sigma_1[T(j\omega)] < l(\omega)$$

The first condition imposes certain performance constraints on the control loop, and the second condition imply certain stability robustness specifications.

Often these constraints can be reformulated as specifications on the singular values of the loop transfer matrix. A more profound statement of these issues can be found in [D2,M1].

One approach to the design under such loop-shape specifications is model-based compensation. In this approach the specifications are satisfied via some target design (full-state or observer design). This design are then recovered to any prescribed degree of accuracy with a loop transfer recovery design - provided that the plant is minimum-phase [D2,A1,S2].

With this procedure robustness constraints imposed on either the plant input or output can be satisfied.

Specific methods of target feedback design are discussed in refs. [A1,B1,S3,S4].

In this paper it will from hereon be assumed that the target design has been performed as a full-state design (i.e. robustness specifications reflected to the plant input node), and the recovery design is therefore an observer-design.

### 3. Eigenstructure design of observers.

The systems considered are the usual minimal state-space systems  $S(A,B,C)$  with  $n$  states and  $m$  inputs/outputs (i.e. a square system). Further it will be assumed that the plant is minimum-phase. The transfer matrix of the plant will be denoted  $P(s)$ .

The (already assigned) full state gain is called  $K$  and the observer gain (to be assigned) is denoted  $F$ . The total model-based compensator is called  $C(s)$ .

The basic idea in eigenspace design of observers is to apply the freedom beyond pole-placement to place the left observer eigenvectors. This is often reasonable since many observer design issues are readily formulated in terms of the left eigenvectors (e.g. suppression of initial estimation errors, filter gain minimization).

Here, it is the purpose to define the subspaces which the left eigenvectors must belong to, and the pole-locations that will guarantee recovery of the full-state loop transfer.

First, however, consider the left observer eigenvectors  $w_i$  of the observer-loop  $A-FC$ :

$$w_i^T (A-FC) = \lambda_i^T \quad w_i = 1, \dots, n \quad (3-1)$$

$\lambda_i$  denotes the  $n$  eigenvalues of  $A-FC$ . After some reordering eq. (3-1) becomes:

$$\begin{aligned} [\lambda_i I - A^T, -C^T] \begin{bmatrix} w_i \\ -F^T w_i \end{bmatrix} &= 0 \quad i=1, \dots, n \\ [\lambda_i I - A^T, -C^T] \begin{bmatrix} w_i \\ z_i \end{bmatrix} &= 0 \quad (3-2) \\ \begin{bmatrix} w_i \\ z_i \end{bmatrix} &\in \text{Ker}[\lambda_i I - A^T, -C^T] \end{aligned}$$

In the following it is assumed that  $(C,A)$  is an observable pair.

This implies that any eigenvalues  $\lambda_i$  can be selected. If  $\lambda_i$  does not belong to the spectrum of  $A$ , eq. (3-2) can be simplified to:

$$w_i^T = z_i^T C \Phi(\lambda_i), \quad \Phi(s) = (sI - A)^{-1} \quad (3-3)$$

This equation shows that  $w_i^T$  must belong to a  $m$ -dimensional subspace defined by  $\lambda_i$ . The specific eigenvectors are determined by the parameter vectors  $z_i$ , consequently eigenvector-selection is equivalent to selecting appropriate vectors  $z_i$ .

Corresponding to  $n$  specific selections of eigenvectors are a filter gain  $F$ . From eq. (3-2) it is found that:

$$\begin{bmatrix} w_1^T \\ \vdots \\ w_n^T \end{bmatrix} = \begin{bmatrix} -z_1^T \\ \vdots \\ -z_n^T \end{bmatrix} \quad (3-4)$$

$$WF = -Z$$

$$F = -W^{-1}Z$$

Clearly a gain  $F$  can only be found if the left eigenvector matrix  $W$  is nonsingular.

Notice that  $F$  is a matrix of real elements if the eigenvalues and parameter vectors are in complex-conjugate pairs.

### 4. LTR with eigenspace techniques

In this section the specific choices of  $z_i^T$  and  $\lambda_i$  which facilitates loop transfer recovery are found.

The derivation is based on the following result from ref. (10).

For a minimal, square and minimum-phase system a full state loop transfer - with the input node as loop breaking point - can be recovered asymptotically if the observer gain selected so that:

- i) the observer-poles are stable
- ii)  $\frac{F(q)}{q} \rightarrow Bu, \quad |\alpha| \neq 0, \quad q \rightarrow \infty$

Here  $q$  is some parameter which  $\lambda_i$  and  $z_i^T$  are functions of. First the eigenvalue-selection is considered. In [K1] it was found that  $p$  eigenvalues approach the zeros of  $S(A,B,C)$ . The remaining  $n-p$  eigenvalues must approach infinity. In the LQG-setting the fast eigenvalues group into  $m$  Butterworth patterns [S1].

Hence the following eigenvalue-selection is adequate:

- i)  $p$  eigenvalues  $\lambda_i$  approach the zeros of  $S(A,B,C)$ .
- ii)  $n-p$  eigenvalues group into  $m$  fast Butterworth patterns.

Let the  $p$  zeros be denoted  $z_{ij}^T$ . The fast poles must be grouped into  $m$  patterns. Let the radii of the  $j$ 'th pattern be  $\lambda_{rj}$  and let the order of the  $j$ 'th pattern be  $l_j$ .

The LTR-problem therefore concerns the selection of  $\lambda_{rj}$ ,  $l_j$  and the vectors  $z_{ij}^T$  ( $j=1, \dots, m$   $i=1, \dots, n$ ).

The condition  $F/q \rightarrow Bu$  restricts the allowable  $z_{ij}^T$ . If the filter gain  $F$  of eq. (3-4) is inserted in this condition:

$$-\frac{W^{-1}Z}{q} \rightarrow Bu, \quad q \rightarrow \infty \quad (4-1)$$

$$-\frac{Z}{q} \rightarrow WBu$$

$$-\frac{z_i^T}{q} \rightarrow z_i^T C \Phi(\lambda_i) Bu \quad i = 1, \dots, n$$

The last equation implies that any  $z_{ij}^T$  must satisfy the limit.

If  $\lambda(q)$  is approaching a zero  $z_i$ , the right hand side of eq. (4-1) can be finite whereas the left hand side converges to zero, consequently  $z_i$  must satisfy:

$$z_i^T C \Phi(z_{oi}) B \alpha \rightarrow 0 \quad i=1, \dots, p \quad (4-2)$$

$$z_i \in \text{Ker} \left[ (C \Phi(z_{oi}) B)^T \right]$$

This value of  $z_i^T$  is denoted a left zero-direction of  $S(A,B,C)^i$  [K1]. Generically the rank of  $C(z_i^T)B$  is  $m-1$ , so  $z_i^T$  must belong to a 1-dimensional subspace. Hence the corresponding left eigenvector  $w_i$  is uniquely determined. The selection of the parameter vectors  $z_i^T$  and  $\lambda_{rj}$  and  $l_j$  associated with fast poles can be divided into three distinct problems. Details of the derivation are found in [S4]. The three cases differs in the geometric properties of the Markov parameters of  $S(A,B,C)$ .

**Uniform rank systems** - Let the Markov parameters of  $S(A, B, C)$  be  $P_i = CA^{i-1}B$ , and:

$$p_i = 0, \quad i=1, \dots, k-1$$

$$|P_k| \neq 0$$

Then LTR is obtained if:

$$1_1 = \dots = 1_n = k$$

The radii  $\lambda_{ij}$  are free to select, and further the parameter vectors  $z_i$  associated with each Butterworth pattern are equal (except for a scaling), but free to select. If the uniform-rank condition is not satisfied the infinite zeros will be of different order. The radii  $\lambda_{ij}$  are still free parameters, but the vectors  $z_i$  are constrained.

Two cases emerge: (Non-uniform-rank - NUR)

**Simple-structure (SS) NUR** - This special class of NUR are defined by:

$$N_L(P_{1_0}) \subset \dots \subset N_L(P_{1_1}) \subset \dots \subset N_L(P_1) \quad (4-3)$$

$$P_i = CA^{i-1}B$$

$$\dim N_L(P_j) = \dim N_L(P_{j-1}) - t_j, \quad j=1, \dots, l.$$

where  $N_i$  denotes left null-space and  $P_i$  are the Markov parameters. The integers  $l_j$  are the orders of the infinite zeros defined by the projected Markov parameters of  $S(A, B, C)$  (these parameters are defined in [K2]).  $t_i$  is number of infinite zeros of order  $l_j$ . The parameter vectors  $z_i$  associated with an infinite zero of order  $l_j$  are given by:

$$z_i^T \in \bigcap_{k=1}^{l_j-1} N_L(P_k) = S_{l_j}, \quad z_i^T P_{l_j} \neq 0 \quad (4-4)$$

All vectors  $z_i^T$  of a Butterworth pattern are equal (except for a scaling).

**Non-simple structure (NSS) NUR** - Any system not characterized by any of the first two classes fall into this group.

The parameter vectors  $z_i^T$  are defined by:

$$z_i^T = \frac{\lambda_i^{-2} x_j}{\lambda_j} \quad \lambda_i \geq 2 \quad (4-5)$$

$$z_i^T \text{CB} \neq 0 \quad l_i = 1$$

The integers  $l$  have the usual interpretation.

The vectors  $x_j^i$  are defined by the left null-space of the Toeplitz matrix:

$$T_k = \begin{bmatrix} CB & & & & & \\ CAB & & & & & 0 \\ & \ddots & & & & \\ & & \ddots & & & \\ & & & \ddots & & \\ CA^{k-1}B & & & & CAB & CB \end{bmatrix} \quad (4-6)$$

$$[x_{1,2} \dots x_0] \in N_L(T_k)$$

For each Butterworth pattern the term  $x_i$  in eq. (4-5, 4-6) is equal for all  $z_i$  (except for a scaling), hence  $z_i$  can be written as:

$$z_i^T = x_0 + z_{iq}^T \quad (4-7)$$

where  $z_i^T$  is a vector-function in  $q$  and  $\lambda_i$ . As  $q \rightarrow \infty$ ,  $z_i^T$  will approach  $x_0$ .

In all three cases it is further required that  $\text{rank}(Z) = m$ , and that the resulting eigenvector-matrix is nonsingular. None of these restrictions are serious, however.

Further note that the conditions here are sufficient conditions (for details on this issue see the Appendix). The scaling mentioned in all 3 cases can be complex, subject only to the complex-conjugate requirements mentioned in § 3.

NSS non-uniform rank systems are associated with systems  $S(A,B,C)$  with a geometrically less transparent left null-space structure of the Markov parameters. However, practical experience indicates that many systems belong to one of the first two categories. Systems with less than 3 inputs and systems with infinite zeros of orders strictly less than 3 are always of simple-structure or of uniform rank. For all 3 cases the parameter vectors  $z_i$  are determined by Markov parameters of  $S(A,B,C)$ . The actual determination of allowable  $z_i$  vectors is a straightforward exercise once the Markov parameters of  $S(A,B,C)$  are known. Once allowable  $z_i^T$  vectors are found the corresponding left eigenvector subspaces can be found.

In all three cases the vectors  $z_i^T$  are not uniquely determined. Hence some extra freedom in selecting left eigenvectors are available. For uniform rank systems  $z_i^T$  are completely free, subject only to weak nonsingularity constraints on  $Z$  and  $W$ . For non-uniform rank systems extra freedom is available for  $z_i^T$  associated with low-order infinite zeros, whereas  $z_i^T$  for the highest order infinite zeros are often uniquely determined.

In all three cases the radii of Butterworth patterns are a free design parameter.

The extra degrees of freedom beyond asymptotic loop transfer recovery can be applied to satisfy secondary design objectives.

Notice that the results are derived under certain conditions. First, it is assumed that the finite zeros are distinct. Secondly, it is assumed that the null-space structure of the projected Markov parameters is simple (i.e. diagonalizable). Further rank-constraints on certain matrices are given in [S4,S6]. However, these constraints are satisfied generically and therefore not mentioned here. For practical purposes none of these assumptions are serious limitations.

Also notice that for SISO-systems the results imply a particularly simple approach to LTR. SISO-systems are of uniform rank, the parameter vectors  $z_i^T$  are scalars and free to select. The left eigenvectors  $w_i$  are then uniquely determined by  $\lambda_i(q)$ . Hence  $F(q)$  is determined only by  $\lambda_i(q)$ .

Finally notice that the requirement of a Butterworth distribution of the fast poles is not strictly necessary. It merely serves as a convenient choice.

As a final remark it is possible - by duality - to state similar results for full state recovery of an observer loop shape (i.e. feedback design based on the output plant node).

## 5. A discussion of eigenstructure-based LTR.

In this section some comments on the applicability of the eigenstructure-based LTR-concepts are provided. These comments may serve as a motivation for dealing with this method. In particular the eigenstructure approach will be compared to the LQG/LTR approach. The comments are to some extent based on quantitative investigations, but mainly based on experience, since closed-form expressions for the applied measures are difficult to obtain in general. In § 6 the comments made here are supported by an example.

**Remark 1** - The eigenstructure-based LTR-design procedure will typically be performed in 2-step manner:

1. Target feedback design which reflects the performance and robustness specifications.
2. Recovery of the target design.

From studies in eigenstructure LTR-design it has been observed that if the fast poles  $\mu$  are manipulated so that

$$|\mu(A-FC)| \approx f \cdot |\beta(A-BK)| \quad (5-1)$$

where  $\beta$  are the dominant eigenvalues of  $A-BK$  and  $2 < f < 4$ , the recovery is often achieved to within

2-3dB over the important band of frequencies, and better roll-off is achieved at high frequencies.

Hence it is not necessary (as often claimed) to move the eigenvalues of the observer into infinity.

**Remark 2** - In order to achieve good recovery it is essential that  $p$  poles of  $A-FC$  and the associated zero-directions are "close" to the zero-structure of  $S(A,B,C)$ . In the eigenstructure LTR-method this objective is achieved in a straightforward manner. In contrast in the LQG/LTR method this is achieved asymptotically (i.e. by decreasing the covariance of the measurement noise). If some zeros are far away from the associated pole of  $A$  it then follows that the measurement noise must be made very small in order to achieve good recovery. One consequence of making the measurement noise covariance very small is that  $F$  will become very large, since  $F \approx qB\alpha$ . This is not desirable since the individual gains of the controller are increased.

In the frequency-domain similar consequences apply. To see this consider the asymptotic value of  $C(s)$ .

$$C_{\text{LIMIT}} = K(s)B \cdot P(s)^{-1} \quad (5-2)$$

The maximum singular value of a typical asymptotic  $C(s)$  is shown in figure 1, where also typical  $\sigma[C(s)]$  curves for finite  $q$ -values are shown. Here it is seen as  $q$  increases the high frequency gains of  $C(s)$  are increased.

Clearly it is not desirable to make  $q$  larger than actually necessary, since the ratio between noise  $n(s)$  and the control signals  $u(s)$  is determined in the high frequency range by:

$$u(s) = -C[I+CP]^{-1} \cdot n(s) \quad (5-3)$$

$$u(s) \approx -C(s) \cdot n(s) \quad , \quad \omega \text{ large}$$

I.e. the noise is amplified into the control signals at high frequencies.

The problem of making  $q$  as small as possible while still achieving reasonable LTR is handled directly in the eigenstructure formulation.

Concerning the fast poles of  $A-FC$  the eigenstructure method allows that the associated zero-directions are assigned directly (i.e. not assigned asymptotically as in the LQG/LTR approach). Again this direct assigned improves the rate of convergence.

In performing these assignments the only remaining design parameters are the recovery parameter  $q$  and the radii of the fast eigenvalues of  $A-FC$ . The latter parameters are discussed next.

**Remark 3.** When one performs an LQG/LTR design it will usually be observed that the fast poles of  $A-FC$  are very uneven distributed - some extremely fast and others less fast. The reason for this is that the poles approach infinity in Butterworth patterns of different orders. One consequence of making some poles much faster than the dominant poles is that the controller gains are increased. The reason for this is that  $F \approx qB\alpha$ , and  $\alpha$  is proportional to the radii of fast poles. To see this it can be found [S4] that for non-uniform rank systems of simple structure  $\alpha$  is:

$$\alpha = \begin{bmatrix} z_1^T & P_{11} \\ \vdots & \vdots \\ z_m^T & P_{1m} \end{bmatrix} \begin{bmatrix} \mu_{r1} & 1 \\ \vdots & \vdots \\ 0 & \vdots \\ \mu_{rm} & 1 \end{bmatrix} \cdot C\alpha, \quad \text{rank}(C\alpha) = m \quad (5-4)$$

where  $\mu$  are the radii of the infinite zeros.

In order to reduce the size of the fast poles the radii can be reduced. This is easily done in the eigenstructure method, since the radii are manipulated directly. Actually the radii can be reduced so much that the size of all fast poles are approximately the same without affecting the recovery process much. In doing so the gains of the controller are reduced (sometimes significantly), since  $\alpha$  is reduced. In the LQG/LTR procedure outlined by Stein & Athans [S2] such changes cannot be handled directly, but must be performed by scaling the input or output variables.

## 6. Examples.

As the first example consider the following minimal system:

$$A = \begin{bmatrix} -30 & 10 & 34.9 & 0 \\ 0 & -40 & 0 & 1 \\ 0 & 0 & -35 & 0 \\ 0 & 0 & 0 & -45 \end{bmatrix} \quad B = \begin{bmatrix} 30 & 0 \\ 35 & 0 \\ -30 & 0 \\ 0 & 50 \end{bmatrix}$$

$$C = \begin{bmatrix} 25 & 0 & 0 & 0 \\ 0 & 25 & 0 & 0 \end{bmatrix}$$

The system has a transmission zeros at  $s = -0.1$ , and it is of non-uniform rank but with simple structure. The Markov parameters are:

$$P_1 = \begin{bmatrix} 750 & 0 \\ 875 & 0 \end{bmatrix} \quad P_2 = \begin{bmatrix} -39925 & 0 \\ -35000 & 1250 \end{bmatrix}$$

According to the rules of § 4 the selection of eigenvalues and parameter values for LTR must be in the following way:

$$\lambda_1 = -0.1, \quad z_1^T C^*(\lambda_1) B = 0 \implies z_1^T = [-0.94837 \quad 0.31718]$$

$$\lambda_2 = \lambda_{r1} q, \quad z_2^T C B \neq 0 \implies z_2^T = [1 \quad 0]$$

$$\lambda_3 = \lambda_{r2} q^{1/2}, \quad z_3^T \in N_L(P_1), \quad z_3^T P_2 \neq 0 \implies z_3^T = [-0.75926 \quad 0.65079]$$

$$\lambda_4 = \bar{\lambda}_3, \quad z_4^T = z_3^T$$

From experiments with different choices of radii it turned out that for

$$q = 3 \cdot 10^{-5}, \quad \lambda_{r1} = -0.0012, \quad \lambda_{r2} = (-1+j)/\sqrt{2}$$

gave LTR to within 1 dB for the minimal singular value of the sensitivity function (for some arbitrary target design), and furthermore the

radii are chosen so that all fast eigenvalues are approximately of the same size and satisfies eq. (5-1) for  $f \approx 4$ .

This design was then compared with an LQG-based LTR design with the same target loop. The weights were selected as:

$$R = BB^T \\ I = I \cdot q$$

as proposed in [S2]. To get approximately the same recovery as for the eigenstructure case  $q$  should be selected as  $q = 10^{-6}$ .

These two designs are now compared with respect to controller gains and recovery convergence.

If the ratio is taken between the individual elements of the observer-gains the following picture arises:

$$|F_{LQGi j} / F_{ESi j}| \approx \begin{bmatrix} 1.5 \cdot 10^3 & 1.4 \cdot 10^5 \\ 1.3 \cdot 10^3 & 10^3 \\ 1.4 \cdot 10^3 & 4 \cdot 10^4 \\ 2 \cdot 8 & 2 \cdot 8 \end{bmatrix}$$

indicating that the  $F_{LQGi}$  gains are several orders of magnitude larger than the  $F_{ES}$  gains. The reason for this is twofold. First the first-order infinite zero is very large, and secondly the second-order infinite zeros are twice as large as for the eigenstructure case. This latter effect is due to the slow convergence towards the zero at  $-0.1$ , which forces  $q$  to larger values in order to have good sensitivity recovery at DC.

If one instead looks at the I/O behaviour of  $C(s)$ , the singular values of  $C(s)$  are shown in figure in figure 2. Again the gains of  $C_{LQGi}$  are larger - in particular at high frequencies as expected from the discussion in § 5. The crossover of  $\sigma[C(s)]$  for the two controllers are respectively  $\omega_{LQGi} \approx 42\text{kHz}$  and  $\omega_{ES} \approx 18\text{kHz}$ .

Finally the maximum singular value of the sensitivity functions are shown in figure 3. Notice that for  $q = 10^{-5}$  the LQG-based recovery is poor at DC. The reason for this is the slow convergence to the zero. For  $q = 10^{-5}$  the associated eigenvalue of  $A - FC$  is  $-0.18$  (almost 100% error), and for  $q = 10^{-6}$  it is  $-0.11$  (10% error). This problem is easily avoided in the eigenstructure formulation.

In the second example the following minimal system is considered to illustrate non-uniform rank systems of non-simple structure:

$$A = \begin{bmatrix} 1 & 2 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 2 & 0 & -1 & 1 & 0 \\ 1 & 0 & -1 & 1 & 2 \\ 2 & 1 & 1 & -1 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

The system has no transmission zeros and it is non-uniform rank with non-simple structure. The system has 2 first order infinite zeros and 1 third-order infinite zero, hence according to § 4 LTR is achieved if eigenvalues and parameter vectors are selected as:

$$\lambda_1 = \lambda_{r1} q, \quad z_1^T \text{CB} \neq 0 \Rightarrow z_1^T = [1 \ 0 \ 0]$$

$$\lambda_2 = \lambda_{r2} q, \quad z_2^T \text{CB} \neq 0, \quad z_2^T z_1 = 0 \Rightarrow z_2^T = [0 \ 1 \ 0]$$

$$\lambda_i = \lambda_{ri} q^{1/3}, \quad z_j^T = \begin{bmatrix} 1 & x_j \\ j=0 & \lambda_i \end{bmatrix}$$

$$[x_1 \ x_0] \begin{bmatrix} \text{CB} & 0 \\ \text{CAB} & \text{CB} \end{bmatrix} = 0$$

$\lambda_{ri} \in$  Butterworth pattern of order 3

With the radii as  $[-1, (-0.5 \pm j\sqrt{3}/2)]$  and  $x_1 = [-2 \ 0 \ 0]$  and  $x_0 = [0 \ 0 \ 1]$ . Now  $z_1^T$  is

$$z_1^T = x_0 + \frac{x_1}{\lambda_{r1}} q^{-1/3} = \left[ -\frac{2}{\lambda_{r1}} q^{-1/3}, 0, 1 \right]$$

Notice here how the vectors are functions of  $q$ . Even the first term is vanishing it cannot be neglected. If it is neglected LTR will not be achieved - whereas the above choice ensures LTR.

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